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# THE PROBLEM OF REALIZING CONSTRAINTS IN DYNAMICS†

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Problems associated with the limiting transition in the second-order Lagrange equations, when the coefficients of rigidity and viscosity and added masses tend to infinity are considered. Under certain conditions, the solutions of the initial equations approach those of the limiting problem with constraints. For integrable constraints, the limiting equations are identical with the usual equations with constraint multipliers. In the case of non-integrable constraints, the solutions depends closely on the way in which they are realized. The generalized models of the dynamics of systems with non-integrable constraints and the properties of the limiting equations of motion are discussed.

**1.** LET $x_1, \ldots, x_n$ be generalized coordinates of a mechanical system, let $T$ be its kinetic energy and $F_1, \ldots, F_n$ generalized forces. If the system is "free" (that is, the coordinates $x$ and velocities $x^\bullet$ are not subject to a non-trivial relation), then its motions can be described by the Lagrange equations

$$[T] = F \tag{1.1}$$

where $[f]$ is the variational derivative $(\partial f/\partial x^\bullet) - \partial f/\partial x$.

If there is a constraint $\Phi(x^\bullet, x, t) = 0$ (in applications, the function $\Phi$ is linear in $x^\bullet$), then Eqs (1.1) are usually replaced by the more general equations

$$[T] = F + \lambda\partial\Phi/\partial x^\bullet, \quad \Phi = 0, \tag{1.2}$$

where $\lambda$ is an as yet undefined multiplier. Let $\partial\Phi/\partial x^\bullet \neq 0$. Then $\lambda$ can be put in the form of an explicit function of $x^\bullet$, $x$ and $t$ without solving (1.2).

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Equations (1.2) are equivalent to the d'Alembert–Lagrange principle:

$$([T]-F)\delta x=0,\quad \Phi=0 \tag{1.3}$$

where the possible displacements $\delta x$ satisfy the equation

$$(\partial\Phi/\partial x^{\bullet})\delta x=0 \tag{1.4}$$

In non-holonomic dynamics, it is usual to start from the d'Alembert–Lagrange principle. Some mathematicians (Gauss, Poinsot, Jacobi, Kirchhoff, etc.) have considered that the d'Alembert–Lagrange principle is independent and has no need of proof (see [1, 2]). However, in the conventional approach to dynamics, this principle is proved with the help of the releasability principle and the axiom of ideal constraints. This principle states that a system with a constraint can be regarded as free, but the external forces $F$ must be supplemented by the constraint reaction

$$R=[T]-F \tag{1.5}$$

The axiom of ideal constraints is expressed by the equation

$$R\delta x=0 \tag{1.6}$$

Relations (1.5) and (1.6), together with the constraint $\Phi=0$ are, of course, equivalent to (1.3). However, Eqs (1.3) on their own without (1.4), which defines possible displacements, do not uniquely define the equations of motion. With this construction of dynamics, therefore, a definition of possible displacements must be included among the axioms. It is independent of axioms (1.5) and (1.6). In fact, in the theory of systems with servocouplings [3], relations (1.5) and (1.6) hold, but the equations for possible displacements are different from (1.4).

Non-holonomic Eqs (1.2) are covariant: all their terms are transformed when the generalized coordinates are replaced by a covariant law. The simple but important covariance property guarantees mathematical consistency of the physical model. An example of the opposite kind is the "Lindelöf model" introduced by Kharlamov [4], in which the motions of the system depend on the way in which the cyclic velocities have been eliminated from the Lagrange function. Here is a simple example: let $L=(x^{\bullet 2}+y^{\bullet 2}+z^{\bullet 2})/2$ be a Lagrange function, and $x^{\bullet}\sin z=y^{\bullet}\cos z$ the equation of a non-integrable constraint. The coordinates $x$ and $y$ are cyclic. Eliminating the cyclic velocity $y^{\bullet}$ and writing the Lagrange equations with Lagrangian $(z^{\bullet 2}+x^{\bullet 2}\cos^{-2}z)/2$, we find that the coordinate $x$ almost always increases without limit as $t$ increases. On the other hand, eliminating the cyclic velocity $x^{\bullet}$, solving the Lagrange equation with Lagrangian $(z^{\bullet 2}+y^{\bullet 2}\sin^{-2}z)/2$ and then integrating relation $x^{\bullet}=y^{\bullet}\,\mathrm{ctg}\,z$, we find that for almost all initial values, the coordinate $x$ is bounded. Hence, the "Lindelöf model" is internally contradictory and so, in general, the question of comparison with experiment should not be raised. We note that Lindelöf himself was not concerned with constructing new models of motion; he made an error in his derivation of the non-holonomic equations from the d'Alembert–Lagrange principle.

The question of the applicability of the non-holonomic model (indeed, like any other model of the mechanics of systems with constraints) cannot, in any specific situation be solved within the framework of an axiomatic scheme without recourse to experimental results. For example, it cannot be stated *a priori* that the non-integrable constraints in the problem of the rolling of a rigid body without slipping are ideal. The point is that, apart from the forces of sliding friction (which do not do work in the rolling of the body), in reality there are always forces of rolling and spinning friction (which generally do work on possible displacements of the rigid body). In fact, we are certain that if there is only one force of Coulomb sliding friction, as the coefficient of dry friction tends to infinity, for any initial data (consistent with the restraints) a rigid body will roll in accordance with the non-holonomic equations. For a uniform billiard ball, this result follows from the classical studies of Euler and Coriolis (see [5]). More-general results on realizing non-holonomic constraints by the forces of Coulomb friction have also been obtained [6, 7].

**2.** The formal-axiomatic method of justifying the dynamics of systems with constraints (which was discussed in Sec. 1) has obvious defects: the origin of the initial axioms [such as the axioms of possible displacements (1.4)] remains unknown, and the limits of applicability of the theoretical model remain undefined. From this standpoint, a "constructive" approach to the theory of systems with restraints, based on an analysis of the physical means of realizing them, would be preferable. The main idea here is to make a limiting transition to the "complete" equations of motion of the free system, when certain physical parameters of the system (the coefficients of rigidity and viscosity, and added masses) tend to infinity.

It should be clearly acknowledged that constraints of this kind do not actually exist in nature: they are introduced in order to simplify the complex physical picture of interaction. The constructive method provides a simplified mathematical model, which gives an adequate description of the motion of a mechanical system on the basis of an analysis of the mechanism of that interaction.

The underlying ideas of the constructive approach have been outlined in the works of Le Cornu, Klein and Prandtl on certain paradoxes of dry friction, discovered by Painlevé (see [8]). It was proposed to replace the rigid bodies of the model problems of Painlevé by elastic bodies with large elastic moduli (which, it must be said, correspond better with reality). When this was done, the effects associated with the non-uniqueness and non-existence of solutions of the equations of motion disappeared. The modulus of elasticity was then made to tend to infinity. As a rule, the motion of a system with absolutely rigid bodies is obtained by a passage to the limit of this kind. In the Painlevé problems, however, no limit exists, suggesting that it is incorrect to use the model of a rigid body in these cases.

Courant formulated a general theory of the realization of holonomic (integrable) constraints using a field of elastic forces with large elastic modulus, subsequently proved in [9].

The problem of realizing non-integrable constraints by means of viscous friction forces was posed by Caratheodory [10]. He examined the problem of a skate sliding along the ice under the operation of an additional force $-Nv$, where $N = \text{const} > 0$, and $v$ is the projection of the velocity of the point of contact perpendicular to the plane of the runner. It was shown [11] that, as $N \to \infty$, the motion of a system such as this tends to that of a skate with a non-holonomic constraint and the velocity of the point of contact lies in the plane of the runner.

The motion of a system on which additional viscous friction forces with dissipative Rayleigh function $N\Phi^2/2$ act has been analysed in [12–14] as an extension of the results in [10, 11] to the multidimensional case. The equations of motion have the form

$$[T] = F - N\Phi\partial\Phi/\partial x^{\cdot} \tag{2.1}$$

If the forces $F$ are potential forces ($F = -\partial V/\partial x$), then $(T+V)^{\bullet} = -N\Phi^2$. Thus, the energy is not dissipated in motions for which $\Phi = 0$. In the general case, however, system (2.1) cannot have motions of this kind for finite values of $N$. The friction defined by the Rayleigh function $N\Phi^2/2$ is frequently referred to as anisotropic.

It turns out that, as $N \to \infty$, the solutions of system (2.1) with fixed initial data in any finite time interval $0 < t < \tau$ tend to the solution of the non-holonomic Eqs (1.2). The initial data cannot satisfy the constraint $\Phi = 0$, and so, owing to the presence of a boundary layer, the convergence of the solutions of (2.1) as $N \to \infty$ to the solutions of (1.2) is not uniform in the interval $(0, \tau)$. This theorem about the passage to the limit can be proved by the methods of the theory of singularly perturbed systems of differential equations.

**3.** The passage to the limit of infinite viscosity is not the only one which reduces the system to motion with a constraint. In [15] an analysis is made of the more-general problem of the motion of a free mechanical system with kinetic energy

$$T_N = T + \alpha N\Phi^2/2, \qquad \alpha = \text{const} \geqslant 0$$

on which, apart from the generalized forces, $F$, anisotropic viscous forces with dissipative Rayleigh function $\beta N\Phi^2/2$, $\beta = \text{const} \geqslant 0$ act. The motion is described by the system of equations

$$[T_N] = F - \beta N\Phi\partial\Phi/\partial x^{\cdot} \tag{3.1}$$

If $\Phi$ is a linear homogeneous form with respect to the velocities, then for all $N \geqslant 0$ the function $T_N$ is a positive definite quadratic form with respect to $x^{\bullet}$. The coefficient $\alpha N$ has the sense of an added mass (or moment of inertia). For large values of $N$, a system with kinetic energy $T_N$ possesses strong anisotropy of the inertial tensor: for motions with velocity of equal magnitude, the kinetic energy of the system depends closely on the direction of motion. A classical example is the problem of the motion of a rigid body in a fluid.

We put $\Phi = [a(x), x^{\bullet}]$ and assume that $a \neq 0$. Suppose that $\alpha > 0$ (the case $\alpha = 0$ has already been considered in Sec. 2).

*Theorem 1.* Let $x_N(t)$ be a solution of (3.1) with initial values

$$x_N(0) = x_0, \quad x^{\cdot}_N(0) = v_0 + w_0/N \tag{3.2}$$

where $[a(x_0), v_0] = 0$ and $x_0$, $v_0$ and $w_0$ are independent of $N$. Then in each finite time interval $0 \leqslant t \leqslant \tau$ the limit

$$\lim_{N \to \infty} x_N(t) = x(t)$$

exists, where the limiting motion $x(t)$ and a certain function $\lambda(t)$ satisfy the system of differential equations

$$[T] = F - \alpha\lambda^{\cdot}\partial\Phi/\partial x^{\cdot} - \alpha\lambda[\Phi] - \beta\lambda\partial\Phi/\partial x^{\cdot}, \quad \Phi = 0 \tag{3.3}$$

and the initial data

$$x(0) = x_0, \quad x^{\cdot}(0) = v_0, \quad \lambda(0) = (a(x_0), w_0)/\alpha \tag{3.4}$$

The mechanical meaning of the multiplier $\lambda$ is clear from the limiting relation:

$$\alpha\lambda(t) = \lim_{N\to\infty} N(a(x_N(t)), x_N^{\cdot}(t)) \tag{3.5}$$

When $\alpha = 0$, Eqs (3.3) are the same as the usual non-holonomic equations (1.2). Suppose that the constraint $\Phi = 0$ is integrable: $\Phi = f^{\bullet}(x)$. Then $[\Phi] = 0$ and Eqs (3.3) will describe the motion of a holonomic system in *redundant* coordinates $x$. However, if the constraint $\Phi = 0$ is non-integrable, when $\alpha \neq 0$ (3.3) differ from (1.2).

Theorem 1 was first established in the special case $\beta = 0$ [16, Part II]. The name "*vako* mechanics" was given to the mathematical model of the motion of systems with constraints based on Eqs (3.3) (in which $\beta = 0$) [16].

The *vako* equations were known to Hertz† [17] in the special case of inertial motion (when $F = 0$) (and in fact they were first obtained by Lagrange in connection with problems of variational calculus). Hertz called the trajectories of *vako* motion geodesic paths, and ordinary non-holonomic trajectories—paths of least curvature. He also noticed the difference between non-holonomic and *vako* trajectories in the billiard-ball problem. Thus, the calculations from Sec. 3 of the paper by Kharlamov [4] add nothing to what Hertz observed. Also, that analysis is incomplete and the conclusions drawn are mistaken: *vako* motion of a billard ball includes all non-holonomic motions (the integration constants $\kappa$, $\epsilon$ merely have to be made equal to zero). The reverse is, of course, not true.

The reason for this effect lies in the fact that, when a uniform sphere rolls, the constraint reactions become zero. In any case, this is a formal comment since (as Theorem 1 makes clear) the *vako* model does not have any direct bearing on the problem of the rolling of a rigid body, as stated clearly in [16, Part III, p. 110].

Hertz considered that real systems with constraints move on paths of least curvature, rather than geodesics. His argument is noteworthy: ". . . a sphere moving in accordance with the principle [Hamilton's principle] would decidedly have the appearance of a living thing, steering its course consciously towards a given goal, whilst a sphere following the laws of nature would give the impression of an inanimate mass spinning steadily towards it" [17, p. 20].

Equations (3.3) are covariant and universal: they can be written for any system with a constraint. The theoretical conditions under which these equations are applicable are given by Theorem 1. For instance, if the constraints arise due to anisotropy of the inertial properties of the system ($\alpha \neq 0$, $\beta = 0$), then from a theoretical point of view it is natural in this case to use the *vako* model, but if they arise from the presence of anisotropic friction ($\alpha = 0$, $\beta \neq 0$), the motion of a system with a constraint can be described in the framework of the classical non-holonomic model (cf. [16, Part III, p. 110]).

It is clear from (3.3) that the non-holonomic and *vako* models are extreme cases of a more general mathematical model of the motion of systems with constraints, which includes the constant $k = \alpha/\beta$ (with the dimensions of time), which has to be found by experiment.

In the case of inertial motion ($F = 0$), the constant $k$ has a simple geometrical meaning: it characterizes the deviation of the geodesic curvature of the trajectories of the system (3.3) from the paths of least curvature (of Hertz), the non-holonomic trajectories. To show this, we will introduce the acceleration vector $w$ with components

$$x_i^{\cdot\cdot} + \sum_{j,l} \Gamma_i^{j,l} x_j^{\cdot} x_l^{\cdot}$$

where $\Gamma_i^{j,l}$ are the Christoffel symbols of the Riemann metric defined by the kinetic energy $T$. Suppose that the non-integrable constraint can be described by the equation $(a, x^{\bullet}) = 0$ and $2T = (A(x)x^{\bullet}, x^{\bullet})$. Eliminating $\lambda^{\bullet}$ from (3.3) with the help of the constraint equation, we obtain the acceleration vector

$$w = \lambda(-\beta A^{-1}a + \alpha A^{-1}c) \tag{3.6}$$

$$c = \frac{(A^{-1}b, a)}{(A^{-1}a, a)}a - b, \quad b = \left(\frac{\partial a}{\partial x} - \left(\frac{\partial a}{\partial x}\right)^T\right)x^{\cdot}$$

Since $(A^{-1}a, c) = 0$, from (3.6) we obtain the geodesic curvature

† Who obviously did not call the equations "*vako*"—see Editor's note on p. 585 (Editor's note).

$$|w|^2 = \lambda^2(\beta^2|a|^2 + \alpha^2|c|^2) \tag{3.7}$$

Here $|w|^2$, $|a|^2$, $|c|^2$ are the squares of the lengths of the vectors $w$, $A^{-1}a$, $A^{-1}c$ in the internal metric of $T$.

We will compare the geodesic curvature of the trajectories of system (3.3) with different values of $k$, but the same initial values of $x$, $x^{\bullet}$ and $\lambda$. Since, when $\alpha = 0$, the value of $\beta\lambda$ can be expressed uniquely in terms of $x$, $x^{\bullet}$ (Sec. 1), it is natural to rewrite (3.7) in the form

$$|w|^2 = (\beta\lambda)^2(|a|^2 + k^2|c|^2) \tag{3.8}$$

where $\beta\lambda$, $|a|$ and $|c|$ are known functions of $x$ and $x^{\bullet}$. If $w_*$ is the acceleration of non-holonomic motion in the same state $(x, x^{\bullet})$, then

$$|w|^2 - |w_*|^2 = (\beta\lambda)^2 k^2 |c|^2$$

For integrable constraints $c = 0$ and so $|w|^2 = |w_*|^2$. In the general case, however, when the constraint is non-integrable, $c \neq 0$. We note that the inequality $|w|^2 \geq |w_*|^2$, which follows from (3.8), represents the principle of least curvature of Gauss and Hertz [17].

**4.** In [4], Kharlamov gave a false account of the meaning of the studies [15, 16], suggesting that they proposed replacing the classical non-holonomic model by the *vako* model. However, this was not done anywhere in those papers. On the contrary, in the very first paper of the cycle [16, Parts I and II], a theorem about the passage to the limit (when $\beta = 0$) is proved and the conditions of applicability of the *vako* model, associated with anisotropy of the inertial tensor of the system, thereby indicated. The main result of Kharlamov's paper [4] is the following: using three specific examples, he points out the difference between the solutions of the non-holonomic and *vako* equations of motion and, on that basis, concludes that the *vako* model is unacceptable. However, first, the difference had already been pointed out by Hertz, Hölder and Suslov (for a modern analysis, see [18]) and, secondly, Kharlamov ignores the physical conditions of the applicability of the *vako* model.

Let us look at these examples in more detail.

The *vako* problem of the rolling of a billiard ball was solved in Sec. 3. However, those calculations bear no relation to real dynamics, since the absence of sliding is taken into account by using viscous or dry friction forces, rather than the effect of added masses.

The problem of a skate sliding on ice is discussed in Sec. 6 from the standpoint of the *vako* model. Here, the non-integrable constraint (the velocity of the point of contact lies in the plane of the runner) is achieved by a lateral force, rather than by anisotropy of the inertial tensor. So the dynamics of this system can be described by classical non-holonomic equations (see [10, 11]). A different physical method of realizing the same non-integrable constraint based on the effect of added masses was pointed out in [16, Part III]. Plane-parallel motion of a rigid body (with a plane of symmetry) in the infinite volume of an ideal fluid was considered in the formulation of Kirchhoff. The kinetic energy of the "body-plus-fluid" system reduces to the form

$$T = (a_1 u^2 + a_2 v^2 + b\omega^2)/2$$

where $u$ and $v$ are the velocity components of some point of the body in fixed space, and $\omega$ is its angular velocity. Owing to the effect of added masses, $a_1 \neq a_2$. It was shown [16, Part III] that, due to the change in shape of the body, the mass $a_2$ can be allowed to tend to infinity, and then $a_1$ and $b$ tend to finite limits. According to Theorem 1 (with $\beta = 0$), as $a_2 \to \infty$ the motion of a body of this kind is subject to the non-integrable constraint $v = 0$ (as in the problem of an ice skate) and described by *vako* (rather than non-holonomic) equations. The "fanciful path of the skate" depicted in Fig. 4 of [4], is one of the trajectories of the limiting hydrodynamic problem.

For large, but finite, values of $a_2$, the velocity $v$ is non-zero in the general case. However, the larger the value of $a_2$, the smaller $v$ is, and the difference from *vako* motion of such a body can be as small as we please. It must be borne in mind that the motion of a real ice skate actually also differs from non-holonomic motion (owing to the resistance to rotation of the skate by the ice).

Finally, Suslov's problem of the rotation of a rigid body about a fixed point, constrained by an untwisting thread, was considered in [4]. According to Suslov, the existence of a thread of this kind gives rise to rotation of a rigid body with a non-holonomic constraint: the projection of the angular velocity on a certain direction $l$ in the body becomes zero. We note first that, in actual fact, the thread in no way prevents rotation of the body about the $l$ axis. Thus, Suslov's realization of the constraint is incorrect. This was apparently first pointed out by G. K. Pozharitskii.

A correct realization of Suslov's non-holonomic constraint was proposed by Vagner [19]. This uses the effect of rolling without slipping and, therefore, (in Vagner's realization) the rotation of the body is described by non-holonomic equations. A different realization of a constraint was proposed in [16, Part II] ($\omega_l = 0$). The body is placed in an ideal fluid, an elliptical plate with centre at the point of suspension is attached to it, and the $l$ axis is directed along the minor axis. If the plate is then lengthened without any change of its area, the associated moment of inertia of the body about the $l$ axis will tend to infinity, and the other associated moments to zero. Hence, according to Theorem 1, in the limit the body rotates in accordance with the *vako* equations and satisfies the constraint $\omega_l = 0$.

Poinsot provides an elegant geometrical representation: an elliptical plate, which is the section of an ellipsoid of inertia by a plane orthogonal to $l$, rolls without slipping on a fixed plane orthogonal to the total kinetic moment of the "body-plus-fluid" system, about the point of attachment. We would re-emphasize that, in spite of Kharlamov's assertion in [4, Introduction, Sec. 6], nowhere in [15, 16] is it suggested that the sliding of a skate on ice and the rotation of a rigid body with an untwisting thread can be described by the *vako* equations.

Kharlamov advances and defends the thesis that when a constraint (mathematically defined by an equation) is introduced, this reflects "an essential of the effect studied".

Thus, according to Kharlamov, the dynamics of a system with constraints is uniquely defined by fixing its inertial properties (kinetic energy), the generalized forces and constraint equations. This thesis is refuted by the *servocoupling* theory of Begen [3]. Servocouplings are realized "actively" with the help of automatically controlled effects, and in the analysis of the motion of mechanical systems with servocouplings, the physical means of realizing them cannot be disregarded. The same applies to the theory of control systems: there are different ways of describing fluttering on the boundaries of a discontinuity, which depend on the switching mechanism [20].

The inapplicability of Kharlamov's thesis is particularly evident in the case of the dynamics of systems with collisions, for he finds that the law of reflection is defined only by an equation of one-sided constraint. However, this is not the case: there are different models of collisions (absolutely elastic and anelastic collisions, Newton's hypothesis, etc.), which depend on the physical properties of the colliding bodies. In fact, a collision is not instantaneous: over a short time interval the bodies are subject to deformation, accompanied by energy dissipation. It can be shown that, with the appropriate consistent transition to infinity of the elastic modulus and the coefficient of viscosity, the motion of a free system approaches that of motion with a collision, the limiting model depending substantially on the relation between the physical parameters of the problem [21].

From this standpoint, the main conclusion of [15, 16] is that, in the "passive" realization of non-integrable constraints, the dynamics of a limiting (simplified) system depends closely on the physical means of realizing the constraints.

**5.** Kharlamov asserts ([4, Sec. 3]) that the multipliers $\lambda$ do not have a mechanical meaning and thus "there are no rational premises for assigning specific initial conditions to them that correspond to the problem". This is untrue. The mechanical meaning of the multiplier $\lambda$ is obvious from (3.5). For instance, in the *vako* modification of the Suslov problem (Sec. 4), $\lambda$ can be interpreted as the kinetic moment of the "body-plus-fluid" system about the $l$ axis. The initial value of $\lambda$ is given by (3.5). In particular, if the initial values $x_0$ and $\dot{x}_0$ satisfy the constraint equation, then $\lambda_0 = 0$.

Thus, the initial value for $\lambda$ can be calculated in the "limiting" problem, when $N \neq \infty$. However, a different appraoch can be used. Knowing the position of the system at two close instants of time, we can find the initial values $x_0$, $\dot{x}_0$ and $\lambda_0$ and thus uniquely identify the required solution. We note that unlike the *vako* model, the boundary-value problem is almost always insoluble in non-holonomic mechanics.

Theorem 1 clearly explains the reason why the classical determinacy principle is violated when $\alpha \neq 0$. In fact, as $N \to \infty$ the iniital values of $x_0$ and $x_0^{\bullet}$ (3.2) are the same, but the limiting motions

$$\lim_{N \to \infty} x_N(t)$$

will be different. In the "limiting" system, a small difference in the initial values of order $N^{-1}$ generates finite differences of the solutions over times $t \sim 1$. The "small causes–large consequences" principle is a fundamental mechanism of the quasi-random behaviour of deterministic dynamic systems (see [22], for example).

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